Universidade de Aveiro Departamento de Matemática
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Um modelo matemático para o desemprego: controlo ótimo com dados reais de Portugal

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Dissertação apresentada à Universidade de Aveiro para cumprimento dos requisitos necessários à obtenção do grau de Mestre em Matemática e Aplicações, realizada sob a orientação científica do Doutor Delfim Fernando Marado Torres, Professor Catedrático do Departamento de Matemática da Universidade de Aveiro.

Dedicado aos meus dois leões.

o júri / the jury

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Palavras Chave

Cálculo das Variações, Controlo Ótimo, Desemprego, Portugal, Políticas estatais .

Resumo

Ao longo desta tese, estudamos a teoria do Cálculo das Variações e do Controlo Óptimo enquanto formulamos e apresentamos alguns exemplos. Depois de uma breve revisão geral da realidade do mercado de trabalho em Portugal, propomos um modelo matemático simples de Controlo Óptimo direccionado para o mercado do desemprego efectuando a comparacão em relação a estudos anteriores. Apesar da simplicidade inerente, afirmamos que o modelo é mais realista e útil do que os disponíveis na literatura. Um caso de estudo com dados reais de Portugal (2004-2016), suporta a nossa afirmação.

Keywords

Calculus of Variations, Optimal Control, Unemployment, Portugal, Governmental policies.

Abstract

During this thesis we study the Calculus of Variations and Optimal Control theory while we present a few examples. After a brief overview of the Portuguese employment market reality we propose a simple Optimal Control model for unemployment and we compare it with previous studies. Despite its inherent simpleness, we claim that the model is more realistic and useful than the ones available in the literature. A case study with real data from Portugal (2004-2016) supports our claim.

The farther backward you can look, the farther forward you can see.
- Winston Churchill

Contents

Chapter 1

Introduction

During this thesis we consider and describe two important fields of Mathematical Optimization: the Calculus of Variations and the Optimal Control. The main goal of both fields is to find the target function that maximizes, or minimizes, a real valued objective function and simultaneously satisfies a system of constraints. The Calculus of Variations tries to determine the extrema functions that optimize a given functional while the Optimal Control is mainly a generalization of the first one that considers actual "controls" within the model specifications. As a sweeping statement, we can consider the Calculus of Variations as a sub-genre of the Optimal Control. On Chapter number two we introduce the Calculus of Variations mathematical formulations and necessary/sufficient conditions of optimality while presenting a few examples solved both algebraically and numerically (on Maple). Chapter three mimics the previous chapter approach but applied to the Optimal Control Theory, this chapter also features the theoretical formulations subsequently applied on our case study. The following chapter number four proposes a simple mathematical optimal control model for unemployment while overviews and establishes a comparison between the new model and other authors previous attempts. We present a case study with real data from Portugal that supports our claim of a model that describes more accurately the employment/unemployment market dynamics. In order to validate and conduct the numerical simulation we used a combination of MatLab and ACADO toolkit procedures which we also present during this chapter. Finally, Chapter number five compiles the discussion and conclusion of this thesis.

Chapter 2

The Calculus of Variations

2.1 Introduction

The origin of the Calculus of Variations can be traced back to the XVII century with the branchistochrome problem (We present the analogous formulations of this problem along this sub-chapter) by John Bernoulli in 1696. Leonhard Euler introduced a general mathematical procedure to find the general solution of variational problems oftenly awarded as the beginning of the Calculus of Variations. His effort was intuitive and required only basic mathematics and geometrical awareness of the variational problem. Lagrange afterwards showed that the resolution of each variational problem can be reduced due to a quite general and powerful analytical manipulation. His techniques eliminated the intuitive approach and the geometrical insight that Euler used. With a more general point of view the Calculus of Variations tries to determine the extrema functions that optimize a given functional.

A functional is an input-output box that receives a function and retrieves a real or complex number. The input to the functional operator J is an arbitrary function $y(x)$ which is called the primary dependent variable or merely the primary variable (see 2.1). Please denote that the functional value output is a scalar.

$$J = J(x, y) = F[y] \tag{2.1}$$

The form $J(x, y)$ explicitly shows the two inputs: x and $y(x)$, in function style. The last form uses square brackets and only shows the function y, so, the dependence $y = f(x)$ is implied.

Oftenly the Functionals depend on the function derivatives (2.2). Here F depends on $y'(x) = dy/dx$:

$$J = J(x, y, y') = J[y] \tag{2.2}$$

The notation $J(x, y, y')$ displays the three inputs, while the square bracketed version implies the dependence on x and $y'(x)$ on the function $y = f(x)$. Functionals may also depend on higher derivatives such as $y''(x) = d^2y/d^x2$. The main goal is to make the functional attain a maximum or minimum value depending on the problem's formulation. That target value is called the extrema of the functional.

Remark 2.1.1 An function extremum of $y(x)$ is a location x^* at which $y(x)$ compiles a minimum, maximum or inflexion, an point within the problem domain: $x \in (a, b)$. At that point, calculus tells us that $y'(x) = 0$, hence $dy = 0$ at x since dx is non-zero.

The central problem of the Calculus of Variations is to find which function (extremal) make the functional stationary with respect to variations of the function? This problem is solved by considering the first variation of the functional, therefore a stationary point of J can be either a maximum, a minimum, or an inflexion point. Such classification conditions depends on the second variation of the functional.

2.2 Euler Equation

Regarded as one of simplest and standard problems from Calculus of Variations, it features a small object that travels between two fixed points under the influence of gravity, therefore the problem's objective is to find the path that minimizes the travelling time (or the fastest path). They also need to satisfy necessary and sufficient conditions that guarantee than such solution present solid minimizers/maximizers depending on the problems formulation. Therefore it's also possible to evaluate qualitative properties of those extermals such as their stability and dependence upon parameters variation.

In order to begin the study of the calculus of variation we present the fundamental problem:

$$
\begin{aligned}
J(y) = \int_0^T & F[t, y(t), y'(t)]dt \\
& \text{Subject to:} \\
& y(0) = A \\
& y(T) = Z
\end{aligned}
\tag{2.3}
$$

As previously stated, the main goal of variational calculus is to choose among a set of admissible y paths (extremal) the one that maximizes or minimizes $J[y]$, which can be an absolute (global) or relative (local) extremum. Since variational calculus is based on classical calculus methods it can only pinpoint local extremes. More specifically the calculation outputs an extreme J value only in comparison with his neighboring y paths. The first-order necessary condition for a local minimum/maximum in the calculus of variations is the Euler equation. So, let $y^*(t)$ be a path that is known as an extremal of $J(t)$ and we try to find some property on $y^*(t)$ that doesn't apply to his neighbouring paths. To achieve this , we need to formulate what we may call a perturbing curve $(p(t))$ in order to create a family of non-extremal neighbouring paths which due to (2.3) they must have certain endpoints (which start at $(0, A)$ and ends at (T, Z)). Those perturbing curves $(p(t))$ should be smooth and pass through the points 0 and T so that:

$$p(0) = p(T) = 0 \tag{2.4}$$

Adding $\epsilon p(t)$ to our extremal $y^*(t)$ we are able to perturb our curve. Please notice that ϵ stands for a very small number with varying magnitude. That varying feature of ϵ will allow us to generate all sorts of neighbouring paths:

$$\begin{aligned} y(t) &= y^*(t) + \epsilon p(t) \\ with \;\; \epsilon \to 0 \;\; and \;\; y(t) &\to y^*(t) \end{aligned} \tag{2.5}$$

Since $y^*(t)$ and $p(t)$ are given curves it is the ϵ value that determines each $y(t)$ path that retrieves also one particular value of $J(t)$. The main idea is to consider J as a function of variable ϵ $(J(\epsilon))$ instead of a functional of the y path. Therefore this manipulation allow us to apply classical calculus methods to the resulting function $J = J(\epsilon)$. Since the curve $y^*(t)$ associated with an $\epsilon = 0$ yields an extremal of J, then we must have:

$$\frac{dJ}{d\epsilon} = 0 \tag{2.6}$$

This is the necessary condition (2.6) that the Euler's equation accomplishes. Nonetheless this formulation isn't operational yet, since it envolves an arbitrary perturbing function $p(t)$ and variable ϵ. In order to work with it we need to develop the previously mentioned Euler Equation.

First, let's express J in terms of ϵ and apply its derivative. Replacing the derivative (2.5) in the objective function (2.3) we obtain:

$$J(\epsilon) = \int_0^T F[t, y^*(t) + \epsilon p(t), y^{*'}(t) + \epsilon p'(t)]dt \tag{2.7}$$

Afterwards, through the Leibniz's rule we differentiate upon the integral sign:

$$\begin{aligned}\frac{dJ}{d\epsilon} = \int_0^T \frac{dF}{d\epsilon} dt &= \int_0^T (\frac{dF}{dy}\frac{dy}{d\epsilon} + \frac{dF}{dy'}\frac{dy'}{d\epsilon})dt \\ &= \int_0^T [F_y p(t) + F_{y'} p'(t)]dt\end{aligned} \tag{2.8}$$

Splitting the integral (2.8) in two different ones and settling (2.6) we achieve the following more structured necessary condition for an extremal:

$$\int_0^T F_y p(t)dt + \int_0^T F_{y'} p'(t)dt = 0 \tag{2.9}$$

We can easily notice that the arbitrary variable ϵ has vanished , but we still need to eliminate de arbitrary perturbing function $p(t)$. To attain that goal we integrate (2.9) by parts using the following formula:

$$\int_{t=\alpha}^{t=b} vdt - vu|_{t=\alpha}^{t=b} - \int_{t=\alpha}^{t=b} udv = 0 \quad [u = u(t), v = v(t)] \tag{2.10}$$

Assuming that $v = F_{y'}$ and $u \equiv p(t)$:

$$\begin{aligned} dv &\equiv \frac{dv}{dt}dt = \frac{dF_{y'}}{dt}dt \\ du &\equiv \frac{du}{dt}dt = p'(t)dt \end{aligned} \tag{2.11}$$

Now, we apply the substitution $\alpha = 0$ and $b = T$ into the expressions (2.11) which yields us:

$$\begin{aligned}\int_0^T F_{y'} p(t)dt = [F_{y'} p(t)]_0^T &- \int_0^T p(t)\frac{d}{dt}F_{y'}dt \\ &= -\int_0^T p(t)\frac{d}{dt}F_{y'}dt\end{aligned} \tag{2.12}$$

Due to the assumption $p(0) = p(T) = 0$ the first presented term to the right of the first equals should vanish. So, binding (2.10) with (2.12) we obtain another version for the extermal necessary condition:

$$\int_0^T p(t)[F_y - \frac{d}{dt}F_{y'}]dt = 0 \tag{2.13}$$

With the previous transformation (2.13) we were able to make $p'(t)$ disappear, but $p(t)$ still remains. Although $p(t)$ enters the expression in an arbitrary way, we can conclude that the condition (2.13) can only be satisfied when the bracketed part is made to vanish for every value of t neutralizing the arbitrary component $p(t)$ since the integral may not be equal to zero for some admissible $p(t)$ perturbation. Subsequently we found a necessary condition for an extremal (Euler Equation) free from arbitrary expressions:

$$F_y - \frac{d}{dt}F_{y'} = 0 \quad for \;\; all \;\; t \in [0, T] \tag{2.14}$$

The Euler equation (2.14) can become clearer with the expansion of the derivative $dF_{y'}/dt$ into a more explicit form. Since F is a function with three arguments (t, y, y'), the partial derivative $F_{y'}$ also should be a function of the same three arguments:

$$\begin{aligned} \frac{dF_{y'}}{dt} &= \frac{dF_{y'}}{dt} + \frac{dF_{y'}}{dy}\frac{dy}{dt} + \frac{dF_{y'}}{dy'}\frac{dy'}{dt} \\ &= F_{ty'} + F_{yy'}y'(t) + F_{y'y'}y''(t) \end{aligned} \tag{2.15}$$

By replacing (2.15) into (2.14) and rearranging the terms we achieve a more explicit version of the Euler Equation:

$$F_{y'y'}y''(t) + F_{yy'}y'(t) + F_{ty'} - F_y = 0 \tag{2.16}$$

The previous expanded expression clearly presents the Euler Formula as a second-order nonlinear differential equation.

2.2.1 Examples

To a more concise understanding about the Euler Equation we will present a few examples:

Example 2.2.1 We intend to find out the extremal for the following extremal:

$$\begin{aligned} \max \quad & J(y) = \int_0^2 (12ty + y'^2)dt \\ \text{s.t.} \quad & y(0) = 0 \\ & y(2) = 8 \end{aligned}$$

Since $F = 12ty + y'^2$ we gather the following derivatives:

$$F_y = 12t \quad F_{y'} = 2y' \quad F_{y'y} = 2 \;\; \text{and} \;\; F_{yy'} = F_{ty'} = 0$$

and the Euler–Lagrange equation (2.16) gives:

$$2y''(t) - 12t = 0 \text{ or } y''(t) = 6t$$

Which after integration it yields:

$$y'(t) = 3t^2 + c_1$$
$$y^*(t) = t^3 + c_1 t + c_2 \quad \text{[General Solution]}$$

The arbitrary constants may also be shifted in order to obtain a definite solution. Settling $t = 0$ in the general solution we gather $y(0) = c2$ and from the initial condition $y(0) = 0$ we logically find out that $c2 = 0$. Following the same train of thought we settle $t = 2$ and the general solution retrieves $y(2) = 8 + 2c1$ and from the final condition $y(8) = 0$ we find out that $c1$ is also zero.

$$y^*(t) = t^3 \quad \text{[Definite solution]}$$

We can confirm the previous results with Maple computing the following code:

```
with(VariationalCalculus);
F := (diff(y(t), t))^2-12*t*y(t);
eqEL := EulerLagrange(F, t, y(t));
```

To solve this equation we execute

```
dsolve({op(eqEL), y(0) = 0, y(2) = 8}, y(t))
```

and then we obtain

```
y(t) = t^3
```

which is the solution that we determined previously.

Example 2.2.2 For the problem

$$\begin{aligned} \max \quad & J(y) = \int_1^5 (3t + (y')^{1/2})dt \\ \text{s.t.} \quad & y(1) = 3 \\ & y(5) = 7 \end{aligned}$$

We have $3t + (y')^{1/2}$ so we gather the following derivatives:

$$F_y = 0 \quad F_{y'} = \frac{1}{2}(y')^{-(1/2)} \quad F_{y'y} = -\frac{1}{4}(y')^{-(3/2)} \quad \text{and} \quad F_{yy'} = F_{ty'} = 0$$

and the Euler–Lagrange equation (2.16) yields:

$$-\frac{1}{4}(y')^{-(3/2)}y''(t) = 0$$

The only way to satisfy this equation is to have a constant first derivative y' that yields $y'' = 0$ to vanish the whole left term. So, we replace y' per $c1$ and after integrating we gather the following solution:

$$y^*(t) = c_1 t + c_2$$

Just like our first example the arbitrary constants may also be replaced to obtain a definite solution.We now settle $t = 1$ in the general solution to obtain $y(1) = c_1 t + c_2$ and from the initial condition $y(1) = 3$ we gather $y(1) = c_1 + c_2 = 3$. We then proceed and settle $t = 5$, the general solution retrieves $y(5) = 5c_1 + c_2$ and from the final condition $y(5) = 7$ we find out that $c_1 = 1$ and $c_2 = 2$. Finally we can present the definite solution:

$$y^*(t) = t + 2 \quad \text{[Definite solution]}$$

We can confirm the previous results with Maple computing the following code:

```
with(VariationalCalculus);
F := (diff(y(t), t))^(1/2)+3*t;
eqEL := EulerLagrange(F, t, y(t));
```

To solve this equation we execute

```
dsolve({op(eqEL)[1], y(1) = 3, y(5) = 7});
```

and then we obtain:

```
y(t) = t+2
```

which corroborates the previous solution.

Example 2.2.3 Now we present a problem with a different type of outcome:

$$\begin{aligned} \max \quad & J(y) = \int_0^5 (t + y^2 + 3y')dt \\ \text{s.t.} \quad & y(0) = 0 \\ & y(5) = 3 \end{aligned}$$

Since $F = t + y^2 + 3y'$ we gather the following derivatives:

$$F_y = 2y \quad F_{y'} = 3$$

and writing the Euler equation (2.15) as $2y - 0$ with the solution:

$$y^*(t) = 0$$

Even though this solution is consistent with our initial condition $y(0) = 0$ it violates the final condition $y(5) = 3$. Upon such scenario we need to conclude that there is no admissible extremal among the set of the continuous curves. We intend to present that a variational problem with fixed endpoints may not contain admissible solutions.

2.3 Second-Order Conditions

On the previous sub-chapter we tried to obtain an extremal of a problem, identify and formulate a condition that clearly states that we are dealing with an extremal function regardlessly if it maximizes or minimizes $J[y]$. To attain that goal and formulate the second-order conditions we also need to check the sufficient conditions based on the concept of concavity or convexity.
The first step to distinguish maximization from minimization is to take the second derivative of $J(\epsilon)$, (therefore, $d^2J/d\epsilon^2$), and apply the following general second-order conditions in calculus necessary and sufficient conditions:

Second-order necessary conditions:

$$\frac{d^2J}{d\epsilon^2} \leq 0 \quad [Maximization\ of\ J]$$

$$\frac{d^2J}{d\epsilon^2} \geq 0 \quad [Minimization\ of\ J]$$

Second-order sufficient conditions:

$$\frac{d^2J}{d\epsilon^2} < 0 \quad [Maximization\ of\ J]$$

$$\frac{d^2J}{d\epsilon^2} > 0 \quad [Minimization\ of\ J]$$

To accomplish the second derivative we differentiate $dJ/d\epsilon$ in (2.8) with respect to ϵ knowing that all partial derivatives of $F(t, y, y')$ are, like F functions of t, y, y', and y, y', are, in turn, both functions of ϵ with derivatives:

$$\frac{dy}{d\epsilon} = p(t) \quad and \quad \frac{dy'}{d\epsilon} = p'(t) \tag{2.17}$$

And we have:

$$\begin{aligned} \frac{d^2J}{d\epsilon^2} = \frac{d}{d\epsilon}(\frac{dV}{d\epsilon}\frac{d}{d\epsilon}) &= \int_0^T [F_y p(t) + F_{y'} p'(t)]dt \\ &= \int_0^T [p(t)\frac{d}{d\epsilon}F_y + p'(t)\frac{d}{d\epsilon}F_{y'}]dt \end{aligned} \tag{2.18}$$

Through (2.17) we gather:

$$\frac{d}{d\epsilon}F_y = F_{yy}\frac{dy}{d\epsilon} + F_{y'y}\frac{dy'}{d\epsilon} = F_{yy}p(t) + F_{y'y}p'(t)$$

and equally:

$$\frac{d}{d\epsilon}Fy' = F_{yy'}p(t) + F_{y'y'}p'(t)$$

after a brief transformation (2.18) turns into:

$$\frac{d^2J}{d\epsilon^2} = \int_0^T [F_{yy}p^2(t) + 2F_{yy'}p(t)p'(t) + F_{y'y'}p^{'2}(t)]dt$$

Theorem 2.3.1 *For the fixed-endpoint problem* (2.7) *, if the function* $F(t, y, y')$ *is concave in the variables* (y, y')*, then the Euler Equation is sufficient for an absolute maximum of* $J[y]$*.*

Theorem 2.3.2 *For the fixed-endpoint problem* (2.7)*, if the function* $F(t, y, y')$ *is convex in the variables* (y, y')*, then the Euler Equation is sufficient for an absolute minimum of* $J[y]$*.*

The author [21] delivers the proof for the previous theorems (2.3.1) and (2.3.2). The function $F(t, y, y')$ is concave in (y, y') only if the pair of distinct points in the domain, $(t, y^*, y^{*'})$ and $(t, y, y^{'})$ we gather:

$$\begin{aligned} &F(t, y, y') - F(t, y^*, y^{*'}) \\ \leq\ &F_y(t, y^*, y^{*'})(y - y^*) + F_{y'}(t, y^*, y^{*'})(y' - y^{*'}) \\ =\ &F_y(t, y^*, y^{*'})\epsilon p(t) + F_{y'}(t, y^*, y^{*'})\epsilon p'(t) \end{aligned} \tag{2.19}$$

On the previous equation $y^*(t)$ denotes the optimal path, while $y(t)$ denotes any other path. Integrating both sides of (2.19) with respect of t over the given interval $[0, T[$, we obtain:

$$\begin{aligned} J[y] - J[y^*] &\leq \epsilon \int_0^T [F_y(t, y^*, y^{*'})p(t) + F_{y'}(t, y^*, y^{*'})p'(t)]dt \\ &= \epsilon \int_0^T p(t)[F_y(t, y^*, y^{*'}) - \frac{d}{dt}F_{y'}(t, y^*, y^{*'})]dt \\ &= 0 \quad [since\ y^*(t)\ satisfies\ the\ Euler\ Equation\ 2.14] \end{aligned} \tag{2.20}$$

We have $J[y] \leq J[y^*]$, where $y(t)$ may refer to any other given path. With the previous formulation we presented $y^*(t)$ as a J-Maximizing path while we demonstrated that the Euler Equation is a sufficient condition given that F is concave. The proof of (2.3.2) is analogous to this one.

2.3.1 The Legendre Condition

The concavity feature achieved on (2.19) is a general or global concept that assumes that the function F is strictly concave/convex. But very oftenly F is not globally concave/convex so we need to settle for some weaker second-order conditions.

The Legendre Condition seeks the necessary second-order conditions for local concavity/convexity, but even though it isn't as strong as a sufficient condition it is very useful and used frequently.

Legendre condition states that $F_{y'y'} \leq 0$ is a necessary condition for a maximum of J, and therefore $F_{y'y'} \geq 0$ for a minimum of J. More formally:

$$\begin{aligned} Maximization\ of\ J[y] &\rightarrow F_{y'y'} \leq 0\ \ for\ all\ t \in [0, T] \\ Minimization\ of\ J[y] &\rightarrow F_{y'y'} \geq 0\ \ for\ all\ t \in [0, T] \end{aligned} \tag{2.21}$$

Chapter 3

Optimal Control

3.1 Introduction

The Optimal Control theory is an extension of the Calculus of Variations, it is appliable for problems which Calculus of Variations is simply not suitable such as those involving constraints on the derivatives of functions sought. In Optimal Control problems , the variables are divided in two classes, state variables and control variables. Generally the movement of the state variables is conducted by differential equations. Before proceeding with this chapter goal of studying Optimal Control theoretical and present a few examples, we would like to take a moment to behold the Optimal Control uncapped potential and prolific research.
Optimal Control presents itself as a very useful tool for optimization problems in a broad range of scientific areas. For example, regarding such an important subject as the health sector , we can enumerate Optimal Control research on HIV/AIDS developed in [4] and [10] , tuberculosis [18], dengue [16] or Ebola with [1]. Economics and general social sciences also have their fair share of Optimal Control research, [7] compiles an unified economic growth approach with a Optimal Control application in order to determine optimal capital accumulation, [22] displays game theory applications. It's also interesting to notice how Optimal Control bind with other methodologies such as Neural Networks [11] , or Agent-Based Model simulations [2]. The extension of Optimal Control applications go as far as we have an optimization problem with variables that display certain dynamics (differential equations) across a period of time (fixed or not fixed) and feature one or more tools (controls) than might be changed in order to maximize or minimize our problem. For example [17] features an Tuberculosis-HIV Optimal Control model that uses the antiretroviral therapy (control) in order to minimize the spectrum of Tuberculosis/HIV infected individuals.

We now proceed this empirical work with a theoretical approach of Optimal Control presenting a basic problem. Afterwards we seek the necessary and sufficient conditions of optimality, and we conclude this chapter presenting the theoretical background of some particular cases that are used in our numerical application on Chapter number four.

3.2 Necessary Conditions

Problem Statement: The basic problem of Optimal Control consists of finding a pair $(\boldsymbol{y}, \boldsymbol{u})$ that solves the following problem:

$$max \;\; J(y,u) = \int_a^b f(t, y(t), u(t))dt \tag{3.1}$$

$$subject\ to \;\; y'(t) = g(t, y(t), u(t)) \tag{3.2}$$

$$a, b, \;\; y(a) = y_0 \ is\ fixed; \;\; y(b)\ is\ free; \tag{3.3}$$

We need to select a piecewise continuous control function $u(t)$ to solve the problem. The functions f and g are assumed as continuously differentiable of three independent arguments, from which none of them is a derivative. The model dynamics processes as the following: the state variable y moves according to the differential equation (3.2) and the control variable u must be a piecewise continuous function of time while influences the objective function (3.1) both directly (through his own value) and indirectly (by influencing the state variable y).
An optimal control problem may also have several control and state variables adding substantial complexity to it.

The simplest Calculus of Variations problem had both of the state variable endpoints fixed, but the simplest Optimal Control is somehow a little bit different since it evolves a free endpoint value from the state variable. Similarly to the Calculus of Variations we seek necessary conditions that corroborate a maximizing solution $u^*(t), y(t), a \leq t \leq b$ to the problem described on (3.1) , (3.2) and (3.3). Since the constraining relation (3.2) must hold at each t over the entire interval $a \leq t \leq b$, so we have a sweeping λ for each t, therefore we have a multiplying function $\lambda(t)$ rather than a single Lagrange multiplier.

For any functions y, u satisfiying simultaneously (3.2) and (3.3) and any continuously differentiable function λ, all defined on $a \leq t \leq b$:

$$
\begin{aligned}
\int_{t_a}^{t_b} f(t, y(t), u(t))dt = \\
\int_{t_a}^{t_b} [f(t, y(t), u(t)) + \lambda(t)g(t, y(t), u(t)) - \lambda(t)x'(t)]dt
\end{aligned} \tag{3.4}
$$

Every coefficient of $\lambda(t)$ must sum to zero if (3.2) is satisfied. Integrating by parts the last terms of (3.4) we have:

$$
-\int_{t_a}^{t_b} \lambda(t)x'(t)dt = -\lambda(t_b)x(t_b) + \lambda(t_a)x(t_a) + \int_{t_a}^{t_b} x(t)\lambda'(t)dt \tag{3.5}
$$

Replacing (3.5) in (3.4) we obtain:

$$
\begin{aligned}
\int_{t_a}^{t_b} f(t, y(t), u(t))dt = \\
\int_{t_a}^{t_b} [f(t, y(t), u(t)) + \lambda(t)g(t, y(t), u(t)) \\
+\lambda'(t)x(t)]dt - \lambda(t_b)x(t_b) + \lambda(t_a)x(t_a)
\end{aligned} \tag{3.6}
$$

To develop the necessary conditions of Calculus of Variations on the previous chapter we build an one parameter family of perturbing curves $p(t)$, $y(t) = y^*(t) + \epsilon p(t)$. We will approach the optimal control the same way, we consider a one-parameter family of comparison controls $u^*(t) + ch(t)$ where $u^*(t)$ is the optimal control, $h(t)$ is some fixed functions, and a is a parameter. Let $x(t, a)$, $t_a \leq t \leq t_b$, denote the state variable generated by (3.2) and (3.3) with the control $u^*(t) + ch(t)$. We assume then that $x(t, a)$ is a smooth function of both of his arguments. Naturally $c = 0$ provides the optimal path $y*$. Hence

$$
y(t, 0) = x^*(t), \quad y(t_a, c) = y_0 \tag{3.7}
$$

Considering the functions u^*, y^* and h all held fixed, the value of (3.1) evaluated along the control function $u^*(t) + ch(t)$ and the following state $x(t, c)$ depend on the single parameter c. So we obtain:

$$
\begin{aligned}
J(c) &= \int_{t_a}^{t_b} f(t, x(t, c), u^*(t) + ch(t))dt \\
&\qquad\qquad \text{Using (3.6)} \\
J(c) &= \int_{t_a}^{t_b} [f(t, x(t, c), u^*(t) + ch(t)) \\
&\quad +\lambda(t)g(t, x(t, c), u^*(t) + ch(t)) \\
&\quad +y(t, c)\lambda'(t)]dt - \lambda(t_b)y(t_b, c) + \lambda(t_a)y(t_a, c)
\end{aligned} \tag{3.8}
$$

Since $u*$ denotes a maximizing control, the function $J(c)$ achieves its maximum when $c = 0$. Hence $J'(0) = 0$. Differentiating with respect to c and evaluating at $c = 0$ yields:

$$J'(0) = \int_{t_a}^{t_b} [(f_y + \lambda g_y + \lambda')x_c + (f_u + \lambda g_u)h]dt - \lambda(t_b)x_c(t_b, 0). \tag{3.9}$$

where the featured f_y, g_y and f_u, g_u represent the partial derivatives of the functions f, g with respect to their second and third arguments, respectively. As y_c features the partial derivative of y with respect to its second argument. Being $c = 0$, the functions are evaluated along $(t, y^*(t), u^*(t))$. Since the last term of (3.8) is independent of c so, $x_b(t_a, c) = 0$ since $x(t_a, c) = y_0$ for every a. Up to this point it was only required that $\lambda(t)$ to be differentiable, but now we need λ to obey to the following linear differential equation:

$$\lambda'(t) = -[f_y(t, y^*, u^*) + \lambda(t)g_y(t, y^*, u^*)], \; with \; \lambda(t_b) = 0 \tag{3.10}$$

With the λ awarded in (3.10), (3.9) provided that

$$\int_{t_a}^{t_b} [f_u(t, y^*, u^*) + \lambda g_u(t, y^*, u^*)]hdt = 0 \tag{3.11}$$

Given an arbitrary function $h(t)$ in particular, it must hold for $h(t) = f_u(t, y^*, u^*) + \lambda g_u(t, y^*, u^*)$, so that

$$\int_{t_a}^{t_b} [f_u(t, y^*, u^*) + \lambda(t)g_u(t, y^*(t), u^*(t))]^2 dt = 0 \tag{3.12}$$

Finally it implies the following necessary condition:

$$f_u(t, y^*, u^*) + \lambda(t)g_u(t, y^*(t), u^*(t)) = 0, \; for \; t_a \leq t \leq t_b \tag{3.13}$$

Briefly, we have shown that if the functions $u^*(t)$ and $y^*(t)$ actually maximize (3.1) with the following restrictions (3.2) and (3.3) then there is a continuously differentiable function $\lambda(t)$ such that u^* , y^* and λ simultaneouly satisfy the state equation (3.14) , the multiplier equation (3.15) and finally the optimality equation (3.16) for $t_a \leq t \leq t_b$:

$$x'(t) - g(t, x(t), u(t)), \quad y(t_a) = y_0 \tag{3.14}$$

$$\lambda'(t) = -[f_y(t, y(t), u(t) + \lambda(t)g_y(t, y(t), u(t))], \quad \lambda(t_b) = 0 \tag{3.15}$$

$$f_u(t, y(t), u(t) + \lambda(t)g_u(t, y(t), u(t)) = 0 \tag{3.16}$$

The device for generating the condition is called the Hamiltonian, which is kind of an extension of solving a nonlinear programming problem by forming the Lagrangian.

$$H(t, y(t), u(t), \lambda(t)) \equiv f(t, y, u) + \lambda g(t, y, u) \tag{3.17}$$

And now:

$$\begin{aligned} dH/du = 0 \ \textit{generates}\ (3.16):\ dH/du = f_u + \lambda g_u = 0; \\ -dH/dy = \lambda' \ \textit{generates}\ (3.15):\ \lambda'(t) = -dH/dy = -(f_y + \lambda g_y); \\ dH/d\lambda = y' \ \textit{recovers}\ (3.14):\ y' = dH/d\lambda = g \end{aligned}$$

Furtherly , we have $y(t_a) = y_a$ and $\lambda(t_b) = 0$. At each t, u is a stationary point of the Hamiltonian for the given values of y and λ.
Similarly to the Variational Calculus approach for a maximization problem it's also necessary that:

$$H_{uu}(t, y^*(t), u^*(t), \lambda) \ \leq \ 0$$

$$\text{and} \ \ u^*(t) \ \ \text{that maximizes} \ \ H(t, y^*(t), u, \lambda(t))$$

Analogously for a minimization problem we have:

$$H_{uu}(t, y^*(t), u^*(t), \lambda) \ \geq \ 0$$

$$\text{and} \ \ u^*(t) \ \ \text{that minimizes} \ \ H(t, y^*(t), u, \lambda(t))$$

In order to represent the preceding theory we will present an example:

Example 3.2.1 We intend to find out the solution for the following problem:

$$\begin{aligned} \max \quad & \int_0^1 (x + u)dt \\ \text{s.t.} \quad & x' = 1 - 2u^2 \\ & x(0) = 1 \end{aligned}$$

We form the Hamiltonian:

$$H(t, x, u, \lambda) = x + u + \lambda(1 - 2u^2)$$

The necessary conditions are the restrictions plus:

$$H_u = 1 - 2\lambda u = 0, \quad H_{uu} = -2\lambda$$
$$\lambda' = -H_x = -1 \quad \lambda(1) = 0$$

Which after integration it yields:

$$\lambda = 1 - t$$
$$\text{So } H_{uu} = -2(1-t) \leq 0 \quad \text{for } 0 \leq t \leq 1 \text{ and}$$

$$u = 1/2\lambda = 1/2(1-t)$$

Replacing in the derivative state equation x' we have:

$$x' = 1 - 1/4(1-t)^2, \quad x(0) = 1$$

Integrating and using the boundary condition we achieve our final solution:

$$x(t) = t - 1/4(1-t) + 5/4,$$
$$\lambda(t) = 1 - t,$$
$$u(t) = 1/2(1-t)$$

3.3 Sufficient Conditions

In Calculus of Variations , the necessary conditions are also sufficient for optimality if the integrand $F(t, y, y')$ is concave (convex) in y. For Optimal Control problems the results are analogous. Let's consider that $f(t, y, u)$ and $g(t, y, u)$ are both differentiable concave functions of y, u in the following problem:

$$max \quad \int_{t_a}^{t_b} f(t, y, u)dt \tag{3.18}$$

$$subject\ to \ \ y'(t) = g(t, y, u) \tag{3.19}$$

$$y(t_a) = y_a\ is\ fixed; \tag{3.20}$$

Suppose that the functions y^*, u^*, λ satisfy the necessary conditions and the constrains (3.19) and (3.20) for all the timeline $t_a \leq t \leq t_b$

$$f_u(t, y, u) + \lambda g_u(t, y, u) = 0, \tag{3.21}$$

$$\lambda' = -f_y(t,y,u) - \lambda g_y(t,y,u) \tag{3.22}$$

$$\lambda(t_b) = 0, \tag{3.23}$$

We also suppose that y and λ are continuous with $\lambda(t) \geq 0$ for all t if $g(t,y,u)$ is nonlinear in y or u. The functions y^* and u^* solve the problem in (3.18), (3.19) and (3.20). Nonetheless, if the functions f and g are both concave y, u then the necessary conditions are also sufficient for optimality if $\lambda(t) \geq 0$ holds.
Let's suppose that y^*, u^* and λ satisfy all of the necessary conditions , and let y, u that satisfy (3.19) and (3.20). f^* and g^* compile the functions evaluated on (t, y^*, u^*). If f and g denote the functions evaluated on other feasible path (t, y, u) we must show that:

$$D = \int_{t_a}^{t_b} (f^* - f)dt \geq 0. \tag{3.24}$$

We already stated that f is a concave function of (y, u) , then we have:

$$f^* - f \geq (y^* - y)f_y^* + (u^* - u)f_u^* \tag{3.25}$$

and:

$$\begin{aligned} D &\geq \int_{t_a}^{t_b} [(y^* - y)f_y^* + (u^* - u)f_u^*]dt \\ &= \int_{t_a}^{t_b} [(y^* - y)(-\lambda g_y^* - \lambda') + (u^* - u)(-\lambda g_u^*)]dt \\ &= \int_{t_a}^{t_b} \lambda[g^* - g - (y^* - x)g_y^* - (u^* - u)g_u^*]dt \\ &\geq 0. \end{aligned} \tag{3.26}$$

If the function g is linear in y, u , then λ may assume any sign. The demonstration holds as long as the last square bracket (3.26) is equal to zero. Finally, if f is concave while g is convex and $\lambda \leq 0$, the necessary conditions will also be sufficient for optimality.

3.4 Fixed Endpoints and Bounded Controls

The following chapter will present our Optimal Control case study for the Portuguese unemployment market, and since we'll present a fixed endpoint bounded control problem we will present the theoretical approach associated with our Optimal Control model specifications.

$$max \quad \int_{t_a}^{t_b} f(t,y,u)dt \tag{3.27}$$

$$subject \ to \ \ y'(t) = g(t,y,u) \tag{3.28}$$

$$y(t_a) = y_a, y(t_b) = y_b \ is \ fixed; \tag{3.29}$$

If u^* denotes an optimal control function and y^* the correspondent optimal path of the state function, generated by replacing $u = u^*$ into (3.28) and solving (3.28) and (3.29). Let u denote another feasible control function and let y^* be the corresponding state function obtained by solving (3.28) and (3.29) with this control. J^* denotes also the maximum value in (3.27):

$$\begin{aligned} J - J^* = \Delta J = \int_{t_a}^{t_b} [f(t,y,u) + \lambda g(t,y,u) + y\lambda \\ -f(t,y^*,u^*) - \lambda g(t,y^*,u^*) - y^*\lambda']dt \end{aligned} \tag{3.30}$$

Replacing the integrand in (3.30) its Taylor series around (t, y^*, u^*):

$$\begin{aligned} \Delta J = \int_{t_a}^{t_b} [(f_y + \lambda g_y + \lambda')(y - y^*) \\ +(f_u + \lambda g_u)(u - u^*)]dt + \int_{t_a}^{t_b} \ldots (Taylor) \end{aligned} \tag{3.31}$$

The partial derivatives of f and g that are evaluated along (t, y^*, u^*). Define:

$$\delta y = y - y^*, \quad \delta u = u - u^*. \tag{3.32}$$

The part of J variation δJ that is linear in δy , δu it is called the first variation of J and it is written:

$$\delta J = \int_{t_a}^{t_b} [(f_y + \lambda g_y + \lambda')\delta y + (f_u + \lambda g_u)\delta u]dt \tag{3.33}$$

If $y*$, u^* are optimal for ((3.27) to (3.29)) there is no modification can improve the value of J. Now, we choose λ to satisfy:

$$\lambda'(t) = -[f_y(t,y^*,u^*) + \lambda(t) g_y(t,y^*,u^*)], \tag{3.34}$$

So that the coefficient of δy in (3.33) will retrieve zero. We need:

$$\delta J = \int_{t_a}^{t_b} [f_u(t,y^*,u^*) + \lambda g_u(t,y^*,u^*)]\delta u dt \leq 0 \tag{3.35}$$

for any arbitrary feasible modification of the control δu. In comparison with the previous statements during this chapter it is worth to notice that now the feasibility includes the requirement that the corresponding modified state variable ends in y_b.
To sum up if y^*, u^* are optimal for the presented problem then it should be a function λ such that y^*, u^* and *lambda* satisfy (3.28), (3.29), (3.34) and finally:

$$f_u(t, y^*, u^*) + \lambda g_u(t, y^*, u^*) = 0 \tag{3.36}$$

Now let's observe the following simple example:

Example 3.4.1 We intend to find out the solution for the following problem:

$$\begin{aligned} \max \quad & \int_0^T u dt \\ \text{s.t.} \quad & y' = u^2 \\ & y(0) = y(T) = 0 \end{aligned}$$

We form the Hamiltonian:

$$H = \lambda_0 u + \lambda u^2$$

$$H_u = \lambda_0 - 2\lambda u = 0$$

A choice of $\lambda_0 = 0$ and $u = 0$ does satisfy the previous restrictions.

As stated previously we will also work with bounded control restrictions. The following problem although quite similar as the previous ones it features a bounded control.

$$max \;\; J = \int_{t_0}^{t_b} f(t, y, u) dt \tag{3.37}$$

$$subject\ to \;\; y'(t) = g(t, y, u), \tag{3.38}$$

$$y(t_a) = y_a, \tag{3.39}$$

$$a \leq u \leq b \tag{3.40}$$

After working (3.38) with a multiplier and integrating by parts, one can compute the variation δJ, the linear part from $J - J^*$.

$$\lambda J = \int_{t_0}^{t_1} [(f_y + \lambda g_y + \lambda')\delta y + (f_u + \lambda g_u)\delta u]dt - \lambda(t_1)\delta y(t_1). \tag{3.41}$$

We choose λ to satisfy:

$$\lambda' = -(f_y + \lambda g_y), \quad \lambda(t_1) = 0, \tag{3.42}$$

So that (3.41) refurbishes to:

$$\delta J = \int_{t_0}^{t_1} (f_u + \lambda g_u)\delta u dt \leq 0 \tag{3.43}$$

If the optimal control selection is at his lower bound a at some time t , then the modified control $a + \delta u$ can't be inferior than a for feasibility, therefore $\delta u \geq 0$ is definitely required. The theory of the upper bound b is analogous to this one.

$$\begin{aligned} \delta u \geq 0 \;\; &when \;\; u = a, \\ \delta u \leq 0 \;\; &when \;\; u = b, \\ \delta u \; is \; unrestricted \;\; &when \;\; a < u < b \end{aligned} \tag{3.44}$$

We need (3.43) to be consistent with (3.44). Therefore u will be chosen that:

$$\begin{aligned} u(t) = a \;\; only \;\; if \;\; & f_u + \lambda g_u \leq 0 \;\; at \;\; t \\ a < u(t) < b \;\; only \;\; if \;\; & f_u + \lambda g_u = 0 \;\; at \;\; t \\ u(t) = b \;\; only \;\; if \;\; & f_u + \lambda g_u \geq 0 \;\; at \;\; t \end{aligned} \tag{3.45}$$

For $u^*(t) = a$, then $\delta u \geq 0$ is required, and $(f_u + \lambda g_u)\delta u \leq 0$ only if $f_u + \lambda g_u \leq 0$. This development is analogous for the upper bound $u^*(t) = b$. It is also worth to notice that the right terms of the (3.45)equation implies the left ones. Similarly to the previous sub-section there must be a function λ such that y^*, u^*, λ satisfy (3.38), (3.39) and (3.45). Let's observe the following Hamiltonian:

$$H = f(t, y, u) + \lambda g(t, y, u) \tag{3.46}$$

Then from (3.38) and (3.42) we have:

$$y' = \frac{dH}{d\lambda}, \quad \lambda' = -\frac{dH}{dy} \tag{3.47}$$

The condition (3.45) can be generated by maximizing H subject to (3.40) which is an ordinary nonlinear programming in u.

Chapter 4

Original Results

4.1 Introduction

Unemployment is an extremely severe social and economic problem, born from the difference between demand and supply of the labor market and sometimes emphasized by population's growth. The unemployed population can be defined as the portion of able citizens that desire to work (known as the active population) but, unfortunately, due to insufficient supply, are deprived from working. Generally, unemployment is a precarious social situation since a portion of the population normally struggles to maintain a minimum welfare and consumption level. Simultaneously, regarding the macroeconomic perspective, higher unemployment rates intensify the pressure on social protection measures, e.g., unemployment subsidies, and, consequently, the associated government expenditure. There are numerous policies that interact directly with a country's level of unemployment or, symmetrically, with the level of employment offered to the citizens. As an example, we have the establishment of a minimum wage and currency devaluation. Here, we propose and investigate a simple new model that describes well the current labor market in Portugal.

Regarding available literature focused on the subject, we go back until 2011, time when Misra and Singh [12] proposed a non-linear mathematical model of unemployment based on Nikolopoulos and Tzanetis previous work of 2003 [15]. More precisely, they suggested a model for housing allocation of a homeless population due to a natural disaster, described by a system of ordinary differential equations with the following three variables: unemployed population, employed individuals, and temporarily workers [12]. They analyse the equilibrium of the model, using the stability theory for differential equations, and perform a few numerical simulations, concluding that the unemployment battle may need immediate measures: they predict that the unemployment rate may rise quickly and, if those high unemployment values are reached, then it might be very difficult to overcome that much bigger problem in the future [12]. Misra and Singh developed further their empirical work, and in 2013 they replaced the temporarily employed variable by newly created vacancies with a delayed feature [13]. Another non-linear mathematical model of unemployment was then proposed by Sîrghi et al. in 2014, based on a differential system with distributed time delays [20]. Moreover, they also considered the unemployment level as a signal to the employers to hire at a lower wage [20]. The split of the vacancies variable into current and government created vacancies, was also a significant modification to the previous Misra and Singh models [12, 13]. Recently, Harding and Neamţu followed also the ideas of Misra and Singh [12, 13], extending further the previous efforts by presenting an unemployment model where job search is open both to native and migrant workers [8]. They also consider two policy approaches: the first that aims to reduce unemployment by observing both past values of unemployment and migration, the second considering the past values of unemployment alone [8]. Similarly, by considering the previously mentioned bibliography, Munoli and Gani (2016) seek to define an optimal control policy to unemployment through two possible measures: government policies, focused at providing jobs directly to unemployed persons, and government policies, aiming the creation of new vacancies [14]. The dynamics [14] of the (un)employment market are defined by three differential equations, referring those to variation on unemployed/employed people and vacancies available, quite similarly to Misra and Singh (2013) [13]. Here, taking into account the Portuguese unemployment reality between 2004 and 2016, we propose a few changes in Munoli and Gani (2016) model [14] and apply optimal control to discuss suitable policies for avoiding unemployment.

4.2 Data analysis and Munoli and Gani's 2016 model

As part of our goal, we collected data from IEFP (Instituto do Emprego e Formação Profissional), a Portuguese government institution focused on providing education and support for the unemployed population [24] and Bank of Portugal [23]. We compiled the number of unemployed persons (U), the unemployment rate (RU) and vacancies available at the IEFP (D), concerning the time period from January 2004 until June 2016 with monthly frequency. A total of 150 observations from each variable were collected, U_t, RU_t, D_t, $t = 1, \ldots, 150$, and three new derived variables were defined:

- employed people (total number)

$$E_t = \frac{U_t(1 - RU_t)}{RU_t}; \tag{4.1}$$

- unemployment change rate

$$RCU_t = \frac{U_t - U_{t-1}}{U_{t-1}}; \tag{4.2}$$

- employment change rate

$$RCE_t = \frac{E_t - E_{t-1}}{E_{t-1}}. \tag{4.3}$$

Formulas (4.2) and (4.3) indicate the rate of change of unemployed and employed people, respectively. We could not obtain the total number of employees in Portugal during our time frame. Therefore, we proceeded with an indirect calculation (4.1) using the variables that we actually could gather: the rate of unemployment RU and the total number of unemployed people U.

Munoli and Gani (2016) [14] present a model that tries to emulate an unemployment environment. Their model consists of three ordinary differential equations, considering the unemployed (U), the employed (E), and available vacancies (V):

$$\begin{cases} \dfrac{dU(t)}{dt} = \Lambda - \kappa U(t)V(t) - \alpha_1 U(t) + \gamma E(t), \\ \dfrac{dE(t)}{dt} = \kappa U(t)V(t) - \alpha_2 E(t) - \gamma E(t), \\ \dfrac{dV(t)}{dt} = \alpha_2 E(t) + \gamma E(t) - \delta V(t) + \phi U(t). \end{cases} \tag{4.4}$$

The parameters and variables of model (4.4) are described on Table 4.1.

Table 4.1: Variables and parameters considered by Munoli and Gani (2016) [14].

Variable	Meaning
$U(t)$	Number of unemployed persons at time t
$E(t)$	Number of employed persons at time t
$V(t)$	Number of vacancies at time t
Λ	Number of unemployed persons that is increasing continuously
κ	Rate at which the unemployed persons are becoming employed
α_1	Rate of migration and death of unemployed persons
α_2	Rate of retirement as well as death of employed persons
γ	Rate of persons who are fired from their jobs
ϕ	Represents the rate of creating new vacancies
δ	Denotes the diminution rate of vacancies due to lack of funds

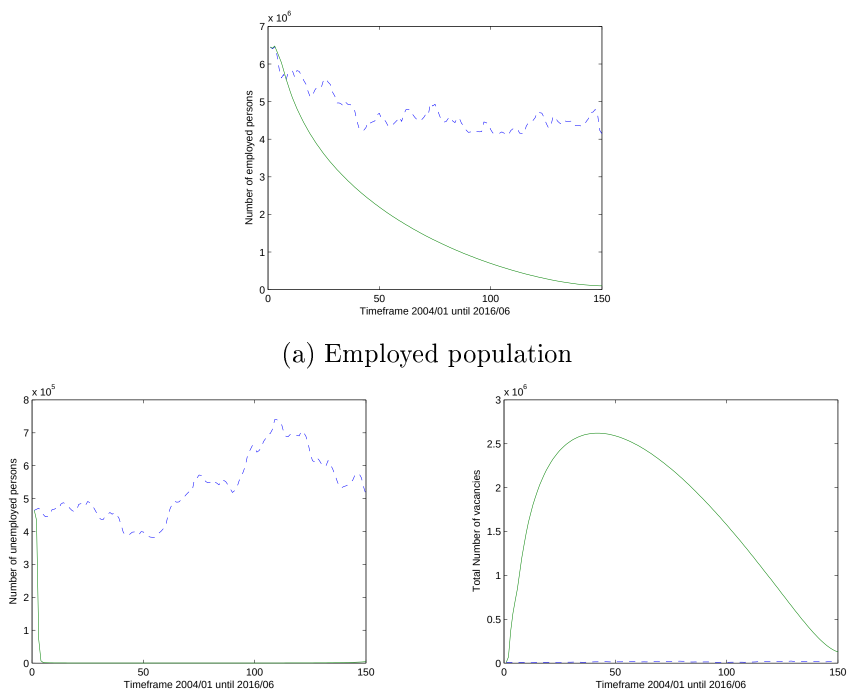
At Munoli and Gani (2016) [14], the initial conditions were given by $U(0) = 10000$, $E(0) = 1000$ and $V(0) = 100$. A time line of 150 units ($t = 150$) was considered [14]. Values of the other parameters were fulfilled according with Table 4.2.

We replaced the initial conditions suggested by the authors of [14] by the ones given by the real data from Portugal. Precisely, we fixed the initial

Table 4.2: Parameters considered by Munoli and Gani (2016) [14].

Parameters	Base Value	Reference
Λ	5000	Misra and Singh (2013) [13]
κ	0.000009	Misra and Singh (2013) [13]
α_1	0.04	Misra and Singh (2013) [13]
α_2	0.05	Misra and Singh (2013) [13]
γ	0.001	Assumed [14]
ϕ	0.007	Assumed [14]
δ	0.05	Assumed [14]

values to the ones of Portugal at January 2004, when the number of unemployed people was 464450 (U), employed was 6450694 (E, according with (4.1)) and the available vacancies at the time were 4848 (V).



(a) Employed population

(b) Unemployed population

(c) Total vacancies

Figure 4.1: Real data in dashed-blue; prediction from Munoli and Gani model of 2016 [14] in continuous-green.

We can easily state that the suggested model from Munoli and Gani (2016) [14] does not replicate properly the real data from Portugal. Regarding the employment and unemployment (see Figures 4.1 a) Employed population and b) Unemployed population, respectively) the model of [14] kind of implode, since the unemployed/employment values dramatically drop until the end of the time period. Munoli and Gani (2016) [14] differential system (4.4) also suggests an exceptional increase on the supply of job vacancies with a smoother decrease until the end of the time period, a statement that is not supported by the Portuguese real data. Actually, the real number of vacancies presents a smooth fluctuation and a shy tendency to increase over time (see Figure 4.1,c) Total Vacancies).

4.3 New unemployment model and simulations

Given the weak results of Section 4.2, our main goal is to create a new mathematical model that explains more accurately the unemployment environment. With this in mind, we proceed with a few changes in Munoli and Gani (2016) [14] model to achieve the desired result. First of all, we consider that the number of vacancies should be an exogenous variable and not given by any specific differential equation, as stated on previous works. The inherent fluctuation and apparent pattern lead us to fit our data in order to achieve a trustworthy representation of this variable. We fit our data with a 3rd degree Fourier function (see Appendix 6.1), obtaining a reasonable goodness of fit (precisely, a R-square of 0.8046, as given in Appendix 6.1). Figure 4.2 shows the fitted function and the actual vacancies data.

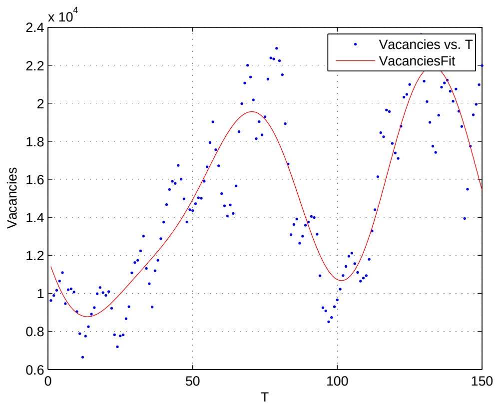


Figure 4.2: Total vacancies. Real data (dots) and fitting by a 3rd degree Fourier function (continuous line).

Having in mind the results of Sîrghi et al. of 2014 [20], we decided to include one variable that compiles the employment created due to the increase of the unemployment. Bigger unemployment rates signals employers to hire at a lower wage and, as a consequence, they are also able to hire more workers. We computed the p-values for Pearson's correlation using a Student's t-distribution for a transformation of the correlation using the `corr(X,Y)` command of MatLab. This function is exact when X and Y are normal. To retrieve such correlation value, we used the two transformed variables RCU_t and RCE_t (formulas (4.2) and (4.3)), obtaining the value of 0.7161. We also found that the constant rate, at which the number of unemployed persons is increasing continuously, as assumed by Munoli and Gani (2016) [14] and Misra and Singh (2013) [13], is quite small regarding the Portuguese population. For this reason, we have increased the value to 90000. We also note that the other crucial point explaining the under-achievement of previous models with respect to Portuguese data (implosion and general shrinkness of the population, see Section 4.2), is the absence of a value of a constant rate at which the number of employed persons is increasing continuously in order to recoup the loss of people within the system. Considering the stated above, we propose here the following model for unemployment, described by a system of two ordinary differential equations:

$$\begin{cases} \dfrac{dU(t)}{dt} = \Lambda - \kappa U(t)V(t) - \alpha_1 U(t) + \gamma E(t), \\ \dfrac{dE(t)}{dt} = \omega + \kappa U(t)V(t) - \alpha_2 E(t) - \gamma E(t) - \delta E(t) + \rho U(t). \end{cases} \tag{4.5}$$

The meaning of the variables and parameters of our model (4.5) is given in Table 4.3. The values used in our simulations are given in Table 4.4.

Table 4.3: Variables and parameters of our mathematical model (4.5).

Variable	Meaning
$U(t)$	Number of unemployed persons at time t
$E(t)$	Number of employed persons at time t
$V(t)$	Total vacancies at time t
Λ	Number of unemployed persons that is increasing continuously
κ	Rate at which the unemployed persons are becoming employed
α_1	Rate of migration and death of unemployed persons
α_2	Rate of retirement or death of employed persons
γ	Rate of persons who are fired from their jobs
ω	Number of employment created and fullfilled
δ	Denotes the diminution rate of vacancies due to lack of funds
ρ	Rate of employment increase due to labor-force wage devaluation

Given the previous model (4.5) and the parameter values of Table 4.4, we carried out the simulation in Matlab (see Appendix 6). Since the required T to smooth our differential equation was 81 observations, we needed to compress our observed real values (150 observations) in order to achieve a graphical comparison (see Figure 4.3).

As we can see, our model fits the observed data much better than the results obtained from previous models available in the literature. Even though there are a few opposite fluctuations, our model achieves a much more steady environment.

Table 4.4: Parameter values considered in our simulation of model (4.5).

Parameters	Base Value	Reference
Λ	90000	Assumed
κ	0.000009	Misra and Singh (2013) [13]
α_1	0.04	Misra and Singh (2013) [13]
α_2	0.05	Misra and Singh (2013) [13]
γ	0.001	Munoli and Gani (2016) [13]
ω	90000	Assumed
δ	0.05	Munoli and Gani (2016) [13]
ρ	0.7161	Assumed (according to variable correlation)

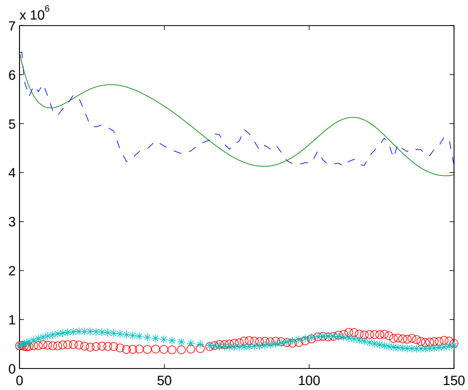


Figure 4.3: Unemployed: real data (circle–red), model simulation (asterisk–light-blue). Employed: real data (dashed–blue), model simulation (continuous–green).

4.3.1 Equilibrium analysis

To compute the equilibrium points of the proposed model (4.5), one needs to solve the following system:

$$\begin{cases} \Lambda - \kappa UV - \alpha_1 U + \gamma E = 0, \\ \omega + \kappa UV - \alpha_2 E - \gamma E - \delta E + \rho U = 0. \end{cases}$$

Direct calculations show that there exists one equilibrium point $E_b = (U^*, E^*)$ only, given by

$$\begin{aligned} E^* &= \frac{\alpha_1 \omega + \Lambda\rho + \kappa(\omega + \Lambda)V}{(\alpha_1 - \rho)\gamma + (\alpha_2 + \delta)\kappa V + \alpha_1(\alpha_2 + \delta)}, \\ U^* &= \frac{\Lambda(\delta + \alpha_2) + (\omega + \Lambda)\gamma}{(\alpha_1 - \rho)\gamma + (\alpha_2 + \delta)\kappa V + \alpha_1(\alpha_2 + \delta)}. \end{aligned} \tag{4.6}$$

Remark 4.3.1 *All the parameters that appear in* (4.6) *are strictly positive.*

Therefore, the numerators of U^* and E^* are always positive. The only possibility for U^* and E^* to be negative would be to have

$$\alpha_1 < \rho \quad \textit{and} \quad (\alpha_2 + \delta)\kappa V + \alpha_1(\alpha_2 + \delta) < |(\alpha_1 - \rho)\gamma|,$$

which is not a reasonable scenario since the variable V, representing the available vacancies, is way bigger than all other parameters appearing in the denominators of U^* and E^*, reasonably valued in the interval $[0, 1]$.

4.3.2 Stability analysis

We now study the local stability of the equilibrium E_b found in Section 4.3.1. To achieve this, we compute the so called variational matrix M of our designated model (4.5):

$$M = \begin{bmatrix} -\kappa V - \alpha_1 & \gamma \\ \kappa V + \rho & -\alpha_2 - \gamma - \delta \end{bmatrix}. \tag{4.7}$$

The characteristic equation associated with our 2×2 matrix (4.7) is

$$\lambda^2 + a_1\lambda + a_2 = 0 \tag{4.8}$$

with

$$\begin{aligned} a_1 &= Vk + \alpha_1 + \alpha_2 + \delta + \gamma, \\ a_2 &= V\alpha_2 k + V\delta k + \alpha_1\alpha_2 + \alpha_1\delta + \alpha_1\gamma - \gamma\rho. \end{aligned} \tag{4.9}$$

The Routh–Hurwitz criterion for second degree polynomials asserts that (4.8) has all the roots in the left half plane (and the system is stable) if and only if both coefficients (4.9) are positive [6]. We just proved the following result.

Theorem 4.3.1 *The equilibrium $E_b = (U^*, E^*)$ given by (4.6) is locally asymptotically stable if and only if*

$$V\alpha_2 k + V\delta k + \alpha_1\alpha_2 + \alpha_1\delta + \alpha_1\gamma > \gamma\rho.$$

Remark 4.3.2 *For the values used to describe the Portuguese reality of unemployment, one has $a_2 = 0.00000090\, V + 0.0033239$, which is strictly positive because V is always non-negative. It follows that the unique equilibrium point of our system is locally asymptotically stable.*

4.4 Optimal Control with real data from Portugal

Portugal is a country with serious unemployment issues during the last decade and the government of Portugal was forced to apply intervention policies in this particular area. Regarding this subject, the adoption of internships and hiring support measures (policies where the government contributes with a share of the worker's salary during a pre-established period, normally one year) have been in force since 1997, with variable magnitude until nowadays. Facing more severe unemployment problems at the 2007/2008 crisis, those measures became quite popular as a fundamental axle regarding the battle against unemployment and integration in the labor market of the recent graduates. With reference to the bibliography on this subject, there are a few empirical works that try to answer or explain the following difficult question: "*Does the supply of internships fight the long-term unemployment?*" The work of Silva et al. (2016) [19] focus on the impact of the internships in the unemployment of graduates compared to other age-similar groups. Another study, Barnwell (2016) [3], addresses the effectiveness of the internship component in the increasing employability of graduates. However, we are not aware of any empirical work that responds concretely to the aforementioned question. Using our representative model of the labor market reality, we now introduce two controls into (4.5):

$$\begin{cases} \dfrac{dU(t)}{dt} = \Lambda - \kappa U(t)V(t)(1+u_2(t)) - \alpha_1 U(t) + \gamma E(t) - u_1(t), \\ \dfrac{dE(t)}{dt} = \omega + \kappa U(t)V(t)(1+u_2(t)) - \alpha_2 E(t) - \gamma E(t) - \delta E(t) + \rho U(t) + u_1(t). \end{cases} \tag{4.10}$$

The first control function, u_1, refers to the unitary supply of internships or support measures; while the second control function, u_2, represents other alternative indirect measures such as lowering corporate tax rates. The offer of an internship, represented by the control measure u_1, has a direct or immediate impact based on the simple premise that an unemployed worker shifts to the employed group due to this incentive. This variable is scaled between -40000 and 40000 because it is possible to add or withdraw internships already operating in the market. The cost of each internship is registered by the monetary value of the support plus all the administrative costs inherent to the planning and execution of the internship. The magnitude of indirect benefits, denoted by the control variable u_2, interacts directly with the exogenous function since those measures affect the natural creation of employment. We settled its value between 0 and 1, being its cost interpreted in the monetary unit (m.u.) of internships. The optimal control problem is as follows:

$$J[U(\cdot), u_1(\cdot), u_2(\cdot)] = \int_0^{150} [A(U(t) - U(0)) + Bu_1(t) + Cu_2(t)]dt \to \min \tag{4.11}$$

subject to the constraints on the control values

$$-40000 \leq u_1(t) \leq 40000, \quad 0 \leq u_2(t) \leq 1, \quad t \in [0, 150], \tag{4.12}$$

the initial conditions

$$U(0) = 464450, \quad E(0) = 6450694, \tag{4.13}$$

the terminal conditions

$$5000000 \leq U(150) + E(150) \leq 8000000 \tag{4.14}$$

and the state constraint

$$\frac{U(t)}{E(t)} \leq 0.13, \quad t \in [0, 150]. \tag{4.15}$$

Since we know that the supply/withdraw of a unitary control u_1 represents a new employed/unemployed in the system, and a government financial cost/gain inherent to this measure, we set B as the reference value equal to 1. In order to settle the value of A, we establish an ideal ratio of 20 to 1, stating that an unemployed person has the similar cost as 19 new employees, using 5% as a target of the utopian level of unemployment. The value of C is set at 40000, representative of 40,000 m.u. expressed in internship currency so that when u_2 is equal to 1 (maximum value) we are stating that we are investing the value/cost of 40000 internships in indirect measures. The constraint (4.15) keeps the unemployment/employment ratio below 13% while (4.14) assures a reasonable level of active labor force between 5 and 8 million people. It is also worth to notice that the problem is unfeasible with a constraint (4.15) below the 0,13 unemployment/employment ratio (using only two decimal places) The initial conditions (4.13) of employment/unemployment level agree with collected data.

We solved the optimal control problem (4.11)–(4.15) using the ACADO Toolkit [9] – see Appendix 6.2. The results are given in Figures 4.4 and 4.5.

As we see, looking to the graphics in Figure 4.4, the model suggests a moderate (about 0.5) adoption of the u_2 control from the beginning of the period until $t = 70$, point at which the number of unemployed people reaches their minimum in our study. At this point it is suggested to switch the selected control: the u_2 control shrinks to zero while a substantial enlarged policy of internships (u_1) is suggested during the time-frame $t = 70$ until approximately $t = 110$. The reason for this policy, during the period $t \in [70, 110]$, might be related to the employment minimum value. Finally, from $t = 110$ until the end of the simulation, we assist to a new rise in unemployment levels. The optimal control approach points out a moderate new supply of indirect incentives u_2 (approximately 0.2) and an internship total contraction, that is, $u_1 = -40000$. The total cost of applying the controls during our time-frame suggests a slight increase in the total investment up to $t = 100$ culminating in a final saving and financial recovery. We note that the application of controls slightly tip the expenditure in approximately 2,000 more internships (monthly) during the 150 month study period.

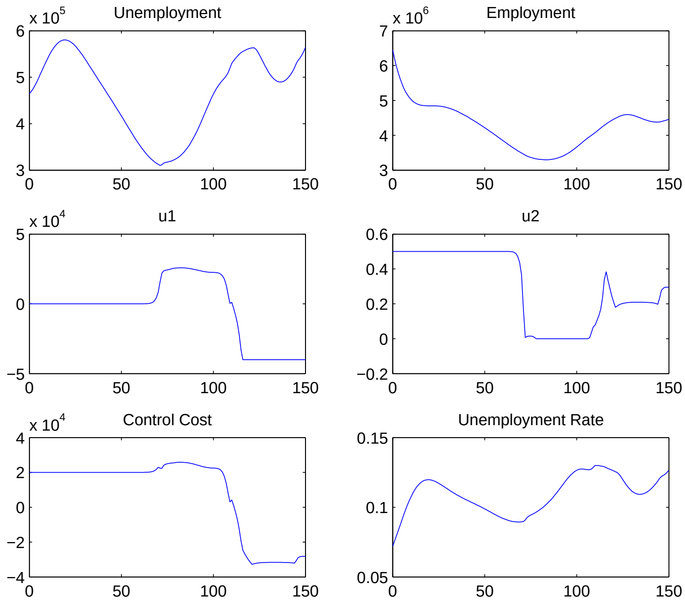


Figure 4.4: Numerical solution to the optimal control problem (4.11)–(4.15) with $A = 20$, $B = 1$ and $C = 40000$.

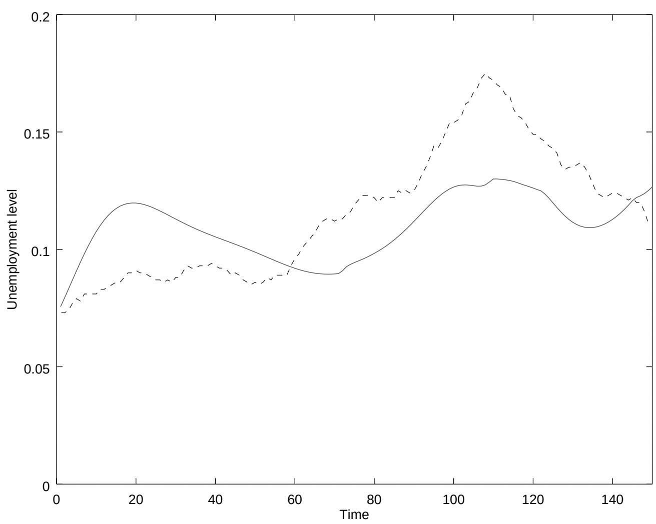


Figure 4.5: Unemployment ratio(U(t)/E(t))): real data (dashed line) versus optimal control simulation (continuous-line).

Finally, comparing the actual data with our optimal control simulation, the level of unemployment regarding the simulation surpasses slightly the real data from the beginning of the simulation until $t = 60$. During the remaining period, the optimal control approach avoids the severe unemployment period from the actual Portuguese data between $t = 100$ and $t = 110$, scoring values above 17%. Overall, and considering the mean over the analysed period, the simulation with optimal controls scores 10.9% slightly better than the real data value of 11.5%, as we see in Figure 4.5.

Chapter 5

Conclusion

In this thesis after the theorical presentation of Calculus of the Variations and Optimal Control fundamentals we propose a simple mathematical model that describes more accurately the real data of unemployment from Portugal in the period 2004-2016. At first, we considered the proposed model from Munoli and Gani (2016) [14], fitting their initial conditions with our data. We found out that the model suggested by Munoli and Gani in 2016 does not describes at all the Portuguese unemployment environment. Because of this, we performed a few changes to the Munoli and Gani model [14], eliminating one of the differential equations, adding an unemployment/wage correlation feature, and adjusting the natural rate of unemployed/employed people in order to control and balance the inner population of the model. Simulations of our model show a much more accurate emulation of the Portuguese unemployment environment. Our results also show a slight decrease on the overall labour force (unemployed plus employed) and, since the Portuguese total population smoothly increased over the last decade, that might signals an higher percentage of inactive population, which may underpin pressure on more social protection measures. Other reading may consist on the premise that the number of unknown unemployed people is increasing and standing apart from the government official records. From the application of optimal control, we can state the following interesting conclusions: the indirect policies should be the predominant method of avoiding unemployment, whereas the supply of internships should be the main choice when the total level of employment offered is low. A possible explanation why to avoid internships in high unemployment periods might be correlated to the severe labour devaluation (considered when we developed the considered model) during these seasons, since the available labour force is already cheap enough and the internships supply might be offering jobs with an increased government expenditure due to the whole administrative costs to people that might turned out to be employed anyway. Through an alternative application of the considered controls, we suggest that such a different approach in periods of severe unemployment may be avoided. As future work, we plan to apply different methods of optimal control to our model in order to seek even more solid ways and tools to control the unemployment issue. For that, we may consider to split the unemployment class into two: unemployed that currently receive welfare from government; and unemployed which do not receive any financial support. These two classes present different government concerns: the first one emphasizes financial pressure and the second one social and well-being pressure. Another interesting study might be to include non-active population, like retired people, and study the optimal control regarding the social security financial health.

Chapter 6

Appendix

6.1 MatLab Code

We provide here our MatLab code for the simulation of the Munoli and Gani (2016) model [14] with Portuguese initial data:

```
f = @(t,x) [5000-0.000009*x(1)*x(3)-0.04*x(1)+0.001*x(2);
0.000009*x(1)*x(3)-0.05*x(2)-0.001*x(2);
0.05*x(2)+0.001*x(2)-0.05*x(3)+0.007*x(1)];
[t,xa] = ode45(f,[0 150],[464450 6450694 9625]);
plot(t,xa(:,1),t,xa(:,2),t,xa(:,3))
xlabel('t'),ylabel('[U,E,V](t)')
```

We obtained a 1737×3 matrix (denoted above by `xa`). Thus, we formatted the space in order to make the comparison with our data (a 150×3 matrix):

```
fxa = linspace(1,1737,150)
form = round(fxa)
nxa = xa(form,:)
```

After getting our new 3×150 matrix (we call it `nxa`), we defined the time frame vector `T`:

```
T = [1:150]
```

The graphical comparison with Munoli and Gani (2016) model [14] is then obtained:

```
plot(T,UnempGlo,T,nxa(:,1))
xlabel('Timeframe 2004/01 until 2016/06'),ylabel('Number of unemployed persons')
plot(T,EmploGlo,T,nxa(:,2))
xlabel('Timeframe 2004/01 until 2016/06'),ylabel('Number of employed persons')
plot(T,VacanciesGlo,T,nxa(:,3))
xlabel('Timeframe 2004/01 until 2016/06'),ylabel('Total Number of vacancies)
General model Fourier3:
f(x) =  a0 + a1*cos(x*w) + b1*sin(x*w) + a2*cos(2*x*w) + b2*sin(2*x*w)
+ a3*cos(3*x*w) + b3*sin(3*x*w)
```

```
Coefficients (with 95% confidence bounds):
a0 =   1.478e+04  (1.444e+04, 1.512e+04)
a1 =       -1262  (-1841, -683.7)
b1 =       -2006  (-2469, -1543)
a2 =       328.2  (-988.4, 1645)
b2 =       -4700  (-5169, -4231)
a3 =       -1992  (-2474, -1510)
b3 =       2.399  (-1202, 1206)
w =     0.04009  (0.03864, 0.04153)

Goodness of fit:
SSE: 5.995e+08
R-square: 0.8046
Adjusted R-square: 0.795
RMSE: 2055
```

```
function [fitresult, gof] = createFit(T, Vacancies)
%CREATEFIT(T,VACANCIES)
%  Create a fit.
%
%  Data for 'VacanciesFit' fit:
%      X Input : T
%      Y Output: Vacancies
%  Output:
%      fitresult : a fit object representing the fit.
%      gof : structure with goodness-of fit info.
%
% Fit: 'VacanciesFit'.
[xData, yData] = prepareCurveData( T, Vacancies );
% Set up fittype and options.
ft = fittype( 'fourier3' );
opts = fitoptions( ft );
opts.Display = 'Off';
opts.Lower = [-Inf -Inf -Inf -Inf -Inf -Inf -Inf -Inf];
opts.StartPoint = [0 0 0 0 0 0 0 0.0421690289072455];
opts.Upper = [Inf Inf Inf Inf Inf Inf Inf Inf];

% Fit model to data.
[fitresult, gof] = fit( xData, yData, ft, opts );

% Plot fit with data.
figure( 'Name', 'VacanciesFit' );
h = plot( fitresult, xData, yData );
legend( h, 'Vacancies vs. T', 'VacanciesFit', 'Location', 'NorthEast' );
% Label axes
xlabel( 'T' );
ylabel( 'Vacancies' );
grid on

corr(RU,RE)
```

6.2 ACADO code

For the numerical solution of the optimal control problem (4.11)–(4.15) described in Section 4.4, we used the ACADO Toolkit, which is a free software environment and algorithm collection for automatic control and dynamic optimization [9]:

```
#include <acado_toolkit.hpp>
#include <acado_gnuplot.hpp>

int main( ){

USING_NAMESPACE_ACADO

DifferentialState    x1,x2,x3 ; the differential states
Control              u1,u2    ;
IntermediateState    mu       ;
the time horizon T

const double t_start = 0.0;
const double t_end   = 150.0;
const double T       = t_end - t_start;
const double a0  =  1.478e+04;
const double a1  =  -1262;
const double b1  =  -2006;
const double a2  =   328.2;
const double b2  =  -4700;
const double a3  =  -1992;
const double b3  =   2.399;
const double w   =   0.04009;

DifferentialEquation f( 0.0, T );

//  -------------------------------------

OCP ocp(t_start,t_end,150);

ocp.minimizeLagrangeTerm( 20*x1 - 20*x1(0) + u1 + 40000*u2 );

mu=a0+a1*cos(x3*w)+b1*sin(x3*w)+a2*cos(2*x3*w)
+b2*sin(2*x3*w)+a3*cos(3*x3*w)+b3*sin(3*x3*w);

f << dot(x1) == 90000-(1+u2)*0.000009*x1*mu-0.04*x1+0.001*x2-u1;
f << dot(x2) == 50000+(1+u2)*0.000009*x1*mu-0.05*x2-0.06*x2-0.001*x2+0.7161*x1+u1;
f << dot(x3) == 1;

ocp.subjectTo( f );  // minimize T s.t. the model
ocp.subjectTo( AT_START, x1 == 464450  );
ocp.subjectTo( AT_START, x2 ==  6450694.0 );
```

```
ocp.subjectTo( AT_START, x3 ==  0 );
ocp.subjectTo( AT_END, 5000000 <= x1 + x2 <=  8000000 );
ocp.subjectTo( -40000 <= u1 <=  40000   );
ocp.subjectTo( 0 <= u2 <=  1   );
ocp.subjectTo( x1/x2 <=  0.13   );

//  ------------------------------------

GnuplotWindow window;
window.addSubplot( x1, "Unemployment" );
window.addSubplot( x2, "Employment"   );
window.addSubplot( u1, "u1"           );
window.addSubplot( u2, "u2" );
window.addSubplot( u1 + 40000*u2 , "Control Cost");
window.addSubplot( x1/x2, "unemployment rate");

OptimizationAlgorithm algorithm(ocp); // the optimization algorithm
algorithm.set( HESSIAN_APPROXIMATION, CONSTANT_HESSIAN );
algorithm.set( KKT_TOLERANCE,1e-2 );
algorithm.set(MAX_NUM_ITERATIONS, 25);
algorithm << window;
algorithm.solve();                   // solves the problem.

algorithm.getDifferentialStates("states.csv");
algorithm.getControls("ctrl.csv");

return 0;
}
```

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